A generalized conditional gradient method for dynamic inverse problems with optimal transport regularization

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Motivation

Motion-aware tomographic reconstruction

Motion on sub-acquisition time scales $\rightsquigarrow$ artefacts in reconstructed images

- Imaging of the lung or heart (motion cannot be suppressed)
- High-resolution imaging (sub-millimeter motion poses problems)

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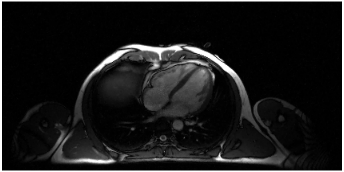

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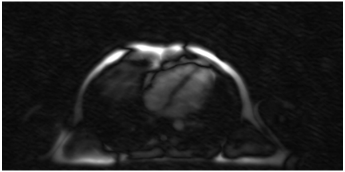

Unregularized reconstruction

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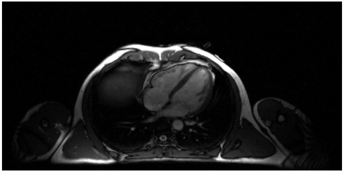

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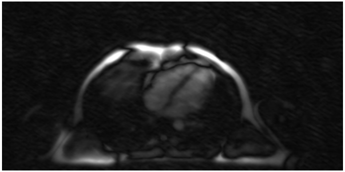
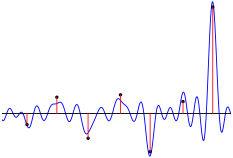


Unregularized reconstruction

$\rightsquigarrow$ **Optimal-transport regularization for for dynamic inverse problems**

Motivation: Sparse recovery

Well-studied problem: Superresolution



- Solve $\mathfrak{F}u = f$ on Σ
- $\mathfrak{F}$ Fourier transform, $\Sigma \subset \mathbb{R}^d$ finite set
- Sparsity assumption: $u = \sum_{i=1}^{N} c_i \delta_{x_i}$

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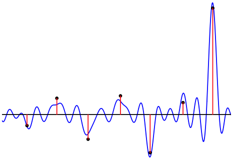

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Radon-norm regularization

- Solve variational problem in space of Radon measures

$$\min_{u \in \mathcal{M}(\Omega)} \|u\|_{\mathcal{M}} \quad \text{subject to} \quad \mathfrak{F}u = f \quad \text{on} \quad \Sigma$$

- Relaxed/regularized version (noisy data)

$$\min_{u \in \mathcal{M}(\Omega)} \tfrac{1}{2}\|\mathfrak{F}u - f\|_{\Sigma,2}^2 + \alpha\|u\|_{\mathcal{M}}$$

[Candès/Fernandez-Granda '13] and many more

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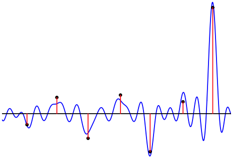

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$\rightsquigarrow$ **Generalization and (fast) recovery algorithms**

Outline

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Static optimal transport

Situation

- $\Omega \subset \mathbf{R}^d$ bounded domain, $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$
- $T : \Omega \to \Omega$ measurable, $\rho_1 = T_{\#}\rho_0$

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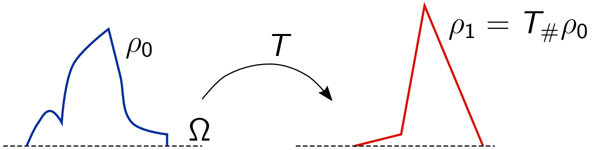

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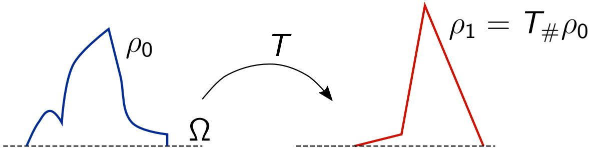

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- Move ρ_0 to ρ_1 in an optimal way
- Cost of moving mass from x to y: $c(x, y) = \frac{1}{2}|x - y|^2$

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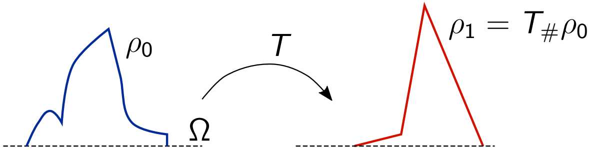

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Optimal transport

Solve $\displaystyle \min_{T:\Omega\to\Omega} \frac{1}{2} \int_\Omega |T(x) - x|^2 \, d\rho_0(x)$ subject to $T_\# \rho_0 = \rho_1$

Dynamic optimal transport

Idea

Introduce a time variable $t \in [0,1]$ and consider evolution of ρ_t

- Time-dependent probability measures

$$t \mapsto \rho_t \in \mathcal{P}(\Omega) \quad \text{for} \quad t \in [0,1]$$

- Velocity field advecting ρ_t

$$v_t \colon [0,1] \times \Omega \to \mathbf{R}^d$$

- (ρ_t, v_t) solves the continuity equation with initial conditions

$$\begin{cases} \partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0 \\ \text{Initial data } \rho_0, \text{ final data } \rho_1 \end{cases} \qquad \text{(CE-IC)}$$

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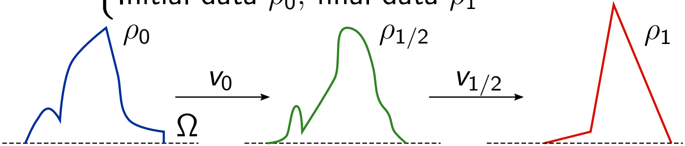

Dynamic optimal transport

Theorem *[Benamou/Brenier '00]*

$$\min_{\substack{(\rho_t, v_t) \\ \text{solving (CE-IC)}}} \int_0^1 \int_\Omega |v_t(x)|^2 \rho_t(x) \, dx \, dt = \min_{\substack{T : \Omega \to \Omega \\ T_\# \rho_0 = \rho_1}} \int_\Omega |T(x) - x|^2 \rho_0(x) \, dx$$

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Advantages of the dynamic formulation

- By introducing $m_t = \rho_t v_t$, we have the convex energy

$$\int_0^1 \int_\Omega |v_t(x)|^2 \rho_t(x)\, dx\, dt = \int_0^1 \int_\Omega \frac{|m_t(x)|^2}{\rho_t(x)}\, dx\, dt$$

- The continuity equation becomes linear

$$\partial_t \rho_t + \operatorname{div} m_t = 0$$

- Full trajectory ρ_t is known and v_t can be recovered from m_t

Optimal transport energy

Definition

- Let $X = (0,1) \times \overline{\Omega}$
- For $(\rho, m) \in \mathcal{M}(X) \times \mathcal{M}(X)^d$, let

$$B(\rho, m) = \int_X \Psi\left(\frac{d\rho}{d\lambda}, \frac{dm}{d\lambda}\right)\, d\lambda$$

where $\lambda \in \mathcal{M}^+(X)$ is such that $\rho, m \ll \lambda$ and

$$\Psi(t, x) = \frac{|x|^2}{2t} \quad \text{if} \quad t > 0, \quad \Psi = \infty \text{ else}$$

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- Generalizes

$$\frac{1}{2} \int_0^1 \int_{\overline{\Omega}} \frac{|m|^2}{\rho} \, \mathrm{d}x \, \mathrm{d}t$$

for functions $\rho : X \to [0, \infty)$, $m : X \to \mathbb{R}^d$, $\mu : X \to \mathbb{R}$ to arbitrary Radon measures

Optimal transport energy

Proposition *[B./Fanzon '20]*

- The functional B is proper, convex, weak* lower semi-continuous and 1-homogeneous

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- The functional B is proper, convex, weak* lower semi-continuous and 1-homogeneous
- If $B(\rho, m) < \infty$ and $\partial_t \rho + \operatorname{div} m = 0$, then
  - $\rho = dt \otimes \rho_t$ for a weak*-continuous curve $t \mapsto \rho_t \in \mathcal{M}^+(\overline{\Omega})$
  - $m = \rho v_t$ for some velocity field $v_t \colon (0, 1) \times \overline{\Omega} \to \mathbf{R}^d$

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$$B(\rho, m) = \frac{1}{2} \int_0^1 \int_{\overline{\Omega}} |v_t(x)|^2 \, d\rho_t(x) \, dt$$

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$\rightsquigarrow$ **Use as an energy for Tikhonov regularization**

Dynamic inverse problem

General setting

- $\Omega \subset \mathbf{R}^d$ bounded open domain, $d \geq 1$
- For $t \in [0, 1]$ assume given
  - H_t Hilbert space (measurement space)
  - $K_t^* : \mathcal{M}(\overline{\Omega}) \to H_t$ linear continuous operator (forward operator)

$\rightsquigarrow$ time dependence allows for spatial undersampling

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Inverse problem

Given some data $\{f_t\}_{t\in[0,1]}$ with $f_t \in H_t$, find a curve of measures $t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$ such that

$$K_t^* \rho_t = f_t \qquad \text{for a.e.} \quad t \in [0, 1]\,. \tag{P}$$

Tikhonov regularization

Inverse problem

Solve $K_t^* \rho_t = f_t$ in H_t for a.e. $t \in [0, 1]$

Tikhonov regularized problem

$$\min_{(\rho,m)\in\mathcal{M}(X)^{d+1}} \underbrace{\frac{1}{2}\int_0^1 \|K_t^*\rho_t - f_t\|_{H_t}^2 \, dt}_{\text{fidelty term}} + \underbrace{\alpha B(\rho, m)}_{\text{optimal-transport term}} + \underbrace{\beta\|\rho\|_{\mathcal{M}}}_{\text{total-variation term}}$$

subject to $\partial_t \rho + \operatorname{div} m = 0$ (CE)

- Regularization parameters $\alpha > 0$, $\beta > 0$

Tikhonov regularization

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Solve $\quad K_t^* \rho_t = f_t \quad \text{in} \quad H_t \quad \text{for a.e. } t \in [0,1]$

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$$\text{subject to} \quad \partial_t \rho + \operatorname{div} m = 0 \qquad \text{(CE)}$$

- Regularization parameters $\alpha > 0$, $\beta > 0$

- (CE) ensures $\rho = \, dt \otimes \rho_t$ and $m \ll \rho$
- $m = v_t \rho_t \rightsquigarrow$ motion field

Dynamic data spaces

Assumption (H)

The spaces H_t vary in a "measurable" way w.r.t $t \in [0, 1]$

- $\exists$ Banach space D and $i_t \colon D \to H_t$ linear continuous
- $i_t(D) \subset H_t$ dense, $\sup_t \|i_t\| \leq C$
- for each $\varphi, \psi \in D$ the map $t \mapsto \langle i_t\varphi, i_t\psi\rangle_{H_t}$ is Lebesgue measurable

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Definition $$L^2_H = \Big\{ f\colon [0,1] \to \cup_t H_t \;\Big|\; f_t \in H_t, \; f \text{ strongly measurable}, \int_0^1 \|f_t\|^2_{H_t} \,\mathrm{d}t < \infty \Big\}$$

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Theorem *[B./Fanzon '20]*
The space L^2_H is a Hilbert space.

Forward operators

Assumption (K)

The operators $K_t^*\colon \mathcal{M}(\overline{\Omega}) \to H_t$ satisfy

- K_t^* linear continuous and weak*-to-weak continuous
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Tikhonov functional

Let $f \in L_H^2$ given data. For $(\rho, m) \in \mathcal{M}(X)^{d+1}$ set

$$T_{\alpha,\beta}(\rho, m) = \frac{1}{2} \int_0^1 \|K_t^* \rho_t - f_t\|_{H_t}^2 \,\mathrm{d}t + \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}}$$

if $\partial_t \rho + \operatorname{div} m = 0$, and $T_{\alpha,\beta}(\rho, m) = \infty$ else

Existence and stability

Assume **(H)**-**(K)**.

Theorem *[B./Fanzon '20]*

$$\min_{(\rho,m)\in\mathcal{M}(X)^{d+1}} T_{\alpha,\beta}(\rho, m) \qquad \text{(Tikh)}$$

admits a solution for $f \in L^2_H$.

- If K_t^* is injective for a.e. t, then the solution is unique.

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Theorem *[B./Fanzon '20]*

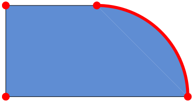
- $\{f^n\}$ noisy data such that $f^n \to f^\dagger$ strongly in L^2_H
- $K_t^* \rho_t^\dagger = f_t^\dagger$ for a.e. $t \in [0,1]$
- (ρ^n, m^n) be a solution to (Tikh) with data f^n and $\alpha_n, \beta_n \to 0$ suitably

Then: $$(\rho^n, m^n) \overset{*}{\rightharpoonup} (\rho^\dagger, m^\dagger) \ \text{ in } \ \mathcal{M}(X)^{d+1}$$

Extremal points of the regularizer

Extremal points

- A $u \in C$ is extremal for a convex set C, if



$$\begin{cases} u = \lambda u_1 + (1-\lambda) u_2 \text{ for } u_1, u_2 \in C, \\ 0 < \lambda < 1 \text{ implies } u = u_1 = u_2 \end{cases}$$

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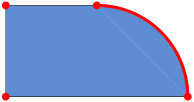

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Goal

- Consider the unit ball of $J_{\alpha,\beta}(\rho, m) = \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}}$
- Determine the extremal points of $J_{\alpha,\beta}$-balls

$$C = \{(\rho, m) \in \mathcal{M}(X) \times \mathcal{M}(X)^d \mid J_{\alpha,\beta}(\rho, m) \leq 1\}$$

Extremal points of the regularizer

Definition

- For $\gamma \in \mathrm{AC}^2([0,1], \overline{\Omega})$ define $\rho_\gamma \in \mathcal{M}(X)$, $m_\gamma \in \mathcal{M}(X)^d$:

$$\rho_\gamma = a_\gamma \, dt \otimes \delta_{\gamma(t)}, \quad m_\gamma = \dot{\gamma}\, \rho_\gamma, \quad a_\gamma^{-1} = \frac{\alpha}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, dt + \beta$$

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Theorem *[B./Carioni/Fanzon/Romero '21]*

The extremal points of C are characterized by

$$\mathrm{Extr}(C) = \{(0,0)\} \cup \mathcal{C}$$

where $\mathcal{C} = \big\{(\rho_\gamma, m_\gamma) \,\big|\, \gamma \in \mathrm{AC}^2([0,1];\overline{\Omega})\big\}$

Sparsity for finite-dimensional data

Fix $N \geq 1$ times $0 < t_1 < t_2 < \cdots < t_N < 1$, let

- H_i finite-dimensional Hilbert space, $\mathcal{H} = \bigtimes_{i=1}^{N} H_i$
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- For $(f_1, \ldots, f_N) \in \mathcal{H}$ find a curve $t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$ such that

$$K_i^* \rho_{t_i} = f_i \quad \text{for} \quad i = 1, \ldots, N$$

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Theorem *[B./Carioni/Fanzon/Romero '21]*

$$\min_{(\rho,m) \in \mathcal{M}(X)^{d+1}} \frac{1}{2} \sum_{i=1}^N \|K_i \rho_{t_i} - f_i\|_{H_i}^2 + \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}(X)}$$

admits a solution of the form $\quad (\rho^*, m^*) = \sum_{i=1}^p c_i \, (\rho_{\gamma_i}, m_{\gamma_i})$

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Proof Also see *[Boyer et al. '19]*, *[B./Carioni '20]*

Outline

A conditional gradient method

Consider the equivalent time-continuous problem

$$\min_{(\rho,m)\in\mathcal{M}(X)^{d+1}} \tilde{T}_{\alpha,\beta}(\rho, m)$$

for $$\tilde{T}_{\alpha,\beta}(\rho, m) = \frac{1}{2}\int_0^1 \|K_t^* \rho_t - f_t\|_{H_t}^2 \, \mathrm{d}t + \varphi(J_{\alpha,\beta}(\rho, m))$$

where, e.g., $\varphi(t) = t + \chi_{\{s \leq M_0\}}(t)$, $M_0 = \frac{1}{2}\int_0^1 \|f_t\|_{H_t}^2 \, dt$

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Conditional gradient method

- Linearization of the smooth term around $(\tilde{\rho}, \tilde{m})$

$$\min_{(\rho,m)\in\mathcal{M}(X)^{d+1}} -\int_0^1 \langle \rho_t, w_t\rangle_{\mathcal{M}(\overline{\Omega})\times C(\overline{\Omega})} \, dt + \varphi(J_{\alpha,\beta}(\rho, m))$$

- $w_t = -K_t(K_t^*\tilde{\rho}_t - f_t)$

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Consider the convex unit ball of $J_{\alpha,\beta}$

$$C = \{(\rho, m) \in \mathcal{M}(X)^{d+1} : \; J_{\alpha,\beta}(\rho, m) \leq 1\}$$

and denote by $\mathrm{Extr}(C) = \{0\} \cup \mathcal{C}$ its extremal points

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- Assume **(H)-(K)**, let $f \in L^2_H$, $t \mapsto \tilde{\rho}_t \in \mathcal{M}(\overline{\Omega})$ weak* continuous, set $w_t = -K_t(K_t^* \tilde{\rho}_t - f_t)$

There exists a solution $(\rho^*, m^*) \in \mathrm{Extr}(C)$ to

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and an $M \geq 0$ such that $(M\rho^*, Mm^*)$ is a solution to

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Let $f \in L^2_H$ be given. Initialize $(\rho^0, m^0) = (0, 0) \in \mathcal{M}(X)^{d+1}$

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2 **Optimization** Solve the quadratic program

$$\bar{c}^n = (\bar{c}_j^n)_j \in \operatorname*{arg\,min}_{c_j^n \geq 0} T_{\alpha,\beta}\left(\sum_j c_j^n (\rho_{\gamma_j^n}, m_{\gamma_j^n})\right)$$

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- Set $(\rho^{n+1}, m^{n+1}) = \sum_j \bar{c}_j^n (\rho_{\gamma_j^n}, m_{\gamma_j^n})$

Convergence

- Define functional distance $r(\rho, m) = T_{\alpha,\beta}(\rho, m) - \min T_{\alpha,\beta}$

Theorem *[B./Carioni/Fanzon/Romero '22]*
Let $f \in L^2_H$, $\alpha, \beta > 0$, $\{(\rho^n, m^n)\}$ in $\mathcal{M}(X)^{d+1}$ the sequence in the conditional gradient method. Then,

- $\{(\rho^n, m^n)\}$ is minimizing with $$r(\rho^n, m^n) \leq \frac{C}{n}$$ where $C > 0$ depends only on f, α, β
- Each weak* accumulation point of $\{(\rho^n, m^n)\}$ is a minimizer for $T_{\alpha,\beta}$

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Improving convergence
Simulations suggest linear convergence. Therefore one could expect that the bound can be improved.

Details and additionals tweaks

- Solve the curve insertion problem

$$\gamma_0^n \in \operatorname*{arg\,min}_{\gamma \in \mathrm{AC}^2([0,1];\overline{\Omega})} -\left(\frac{\alpha}{2}\int_0^1 |\dot{\gamma}(t)|^2 \, dt + \beta\right)^{-1} \int_0^1 w_t^n(\gamma(t)) \, dt$$

  via gradient descent with suitable stepsize rule

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Under suitable regularity assumptions, the gradient descent procedure converges subsequentially to stationary points and strongly in $\mathrm{AC}^2([0,1];\overline{\Omega})$.

- Multiple starts with suitable initial guess (crossovers, random curves, etc.) to increase chance to obtain global minimizer
- Multiple insertion $\rightsquigarrow$ insert all obtained stationary points

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- **Alternative** Curve insertion via dynamic programming
  $\rightsquigarrow$ *[Duval/Tovey '21]*

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- **Sliding step** Perform gradient descent steps for

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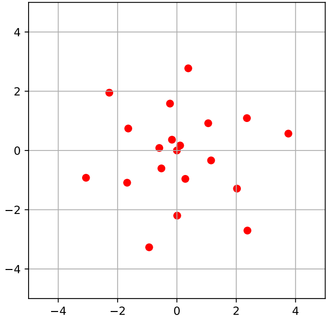
- **Stopping criterion**

$$\left(\frac{\alpha}{2}\int_0^1 |\dot{\gamma}_0^n(t)|^2\,dt + \beta\right)^{-1}\int_0^1 w_t^n(\gamma_0^n(t))\,dt \leq 1$$

or up to some tolerance

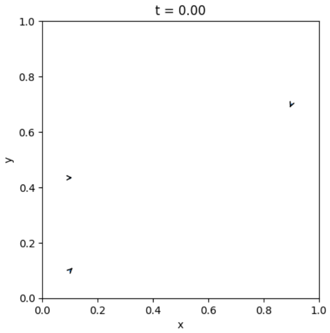
Numerical experiments

- $\Omega = (0,1)^2$, $\sigma = \mathcal{H}^0 \llcorner s$ where $s =$ spiral points in Ω
- $H_t = L^2_\sigma(\mathbf{R}^2, \mathbf{C})$ (time independent)
- $K_t^* : \mathcal{M}(\overline{\Omega}) \to H_t$ masked Fourier transform

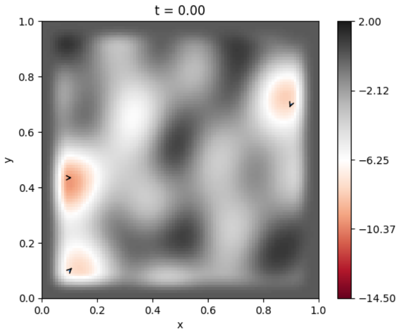


Numerical experiments

A simple example



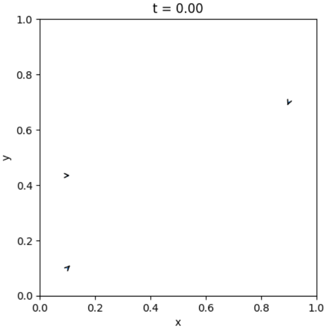
Ground truth



Backprojected data

Numerical experiments

A simple example



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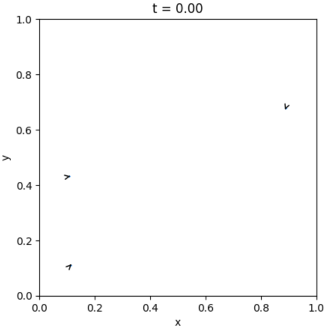
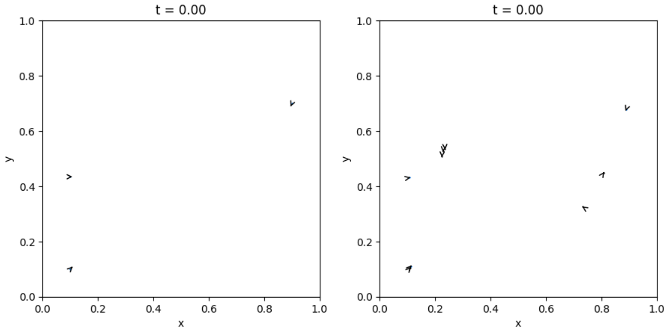


Reconstruction
(thresholded at 0.01)

Numerical experiments

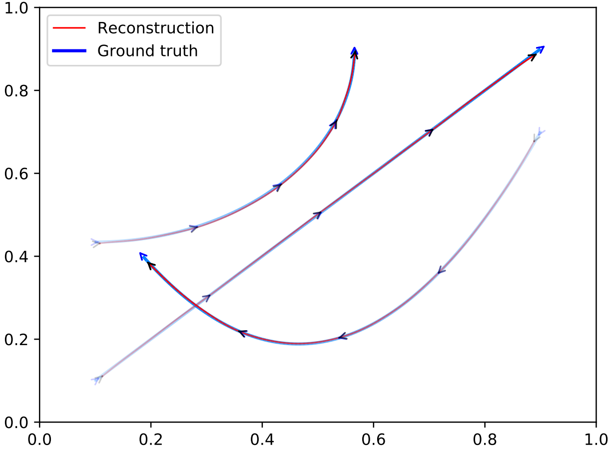
A simple example



Ground truth

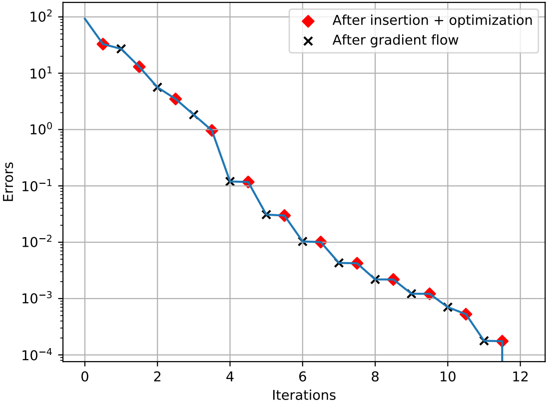
Reconstruction (no thresholding)

Numerical experiments



Reconstructed trajectories

Numerical experiments

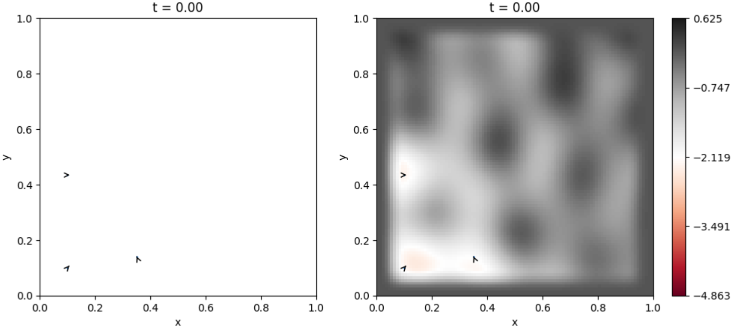


Convergence plot: exhibits linear rate

$$\text{Error} = T_{\alpha,\beta}(\rho^n, m^n) - T_{\alpha,\beta}(\rho^{n+1}, m^{n+1})$$

Numerical experiments

A more difficult example



Ground truth | Backprojected data

Numerical experiments

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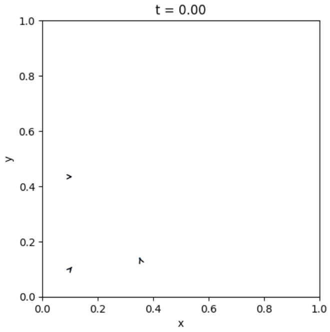
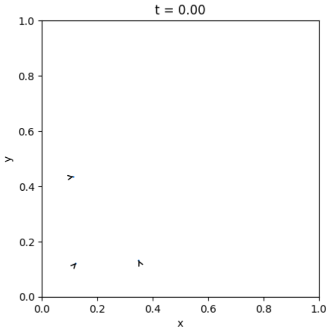


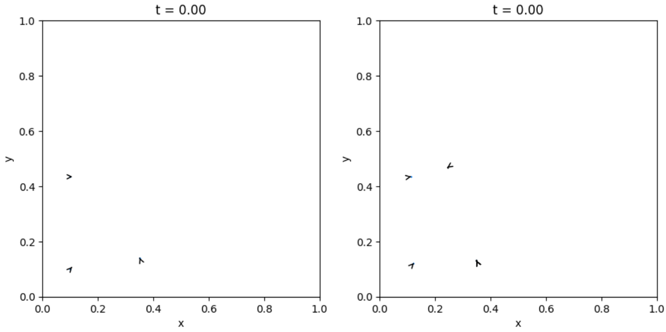
Ground truth



Reconstruction (thresholded at 0.05)

Numerical experiments

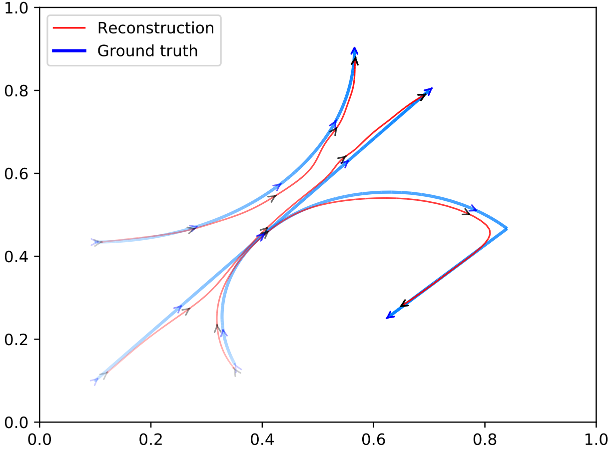
A more difficult example



Ground truth

Reconstruction (no thresholding)

Numerical experiments



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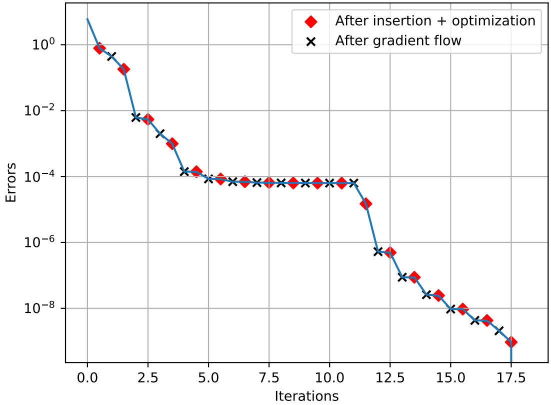
Numerical experiments



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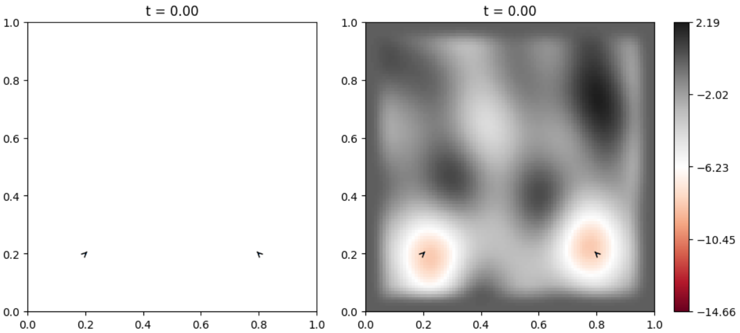
Numerical experiments

A crossing example



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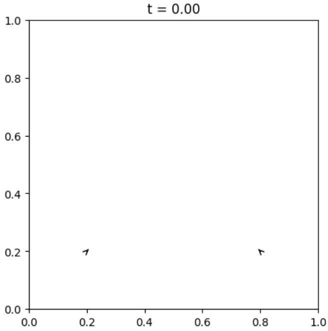

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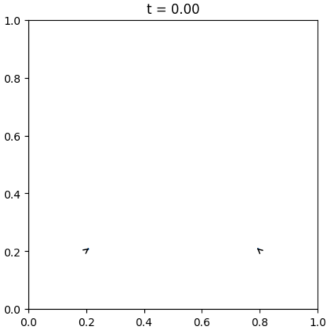

Reconstruction
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Numerical experiments

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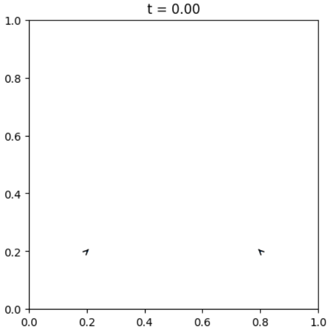

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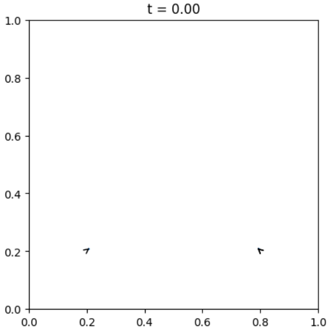

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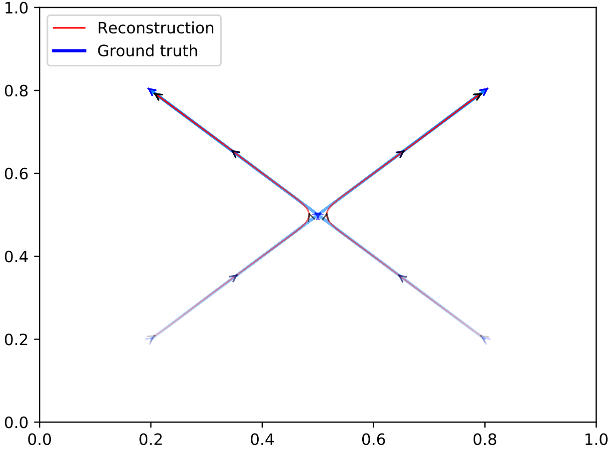

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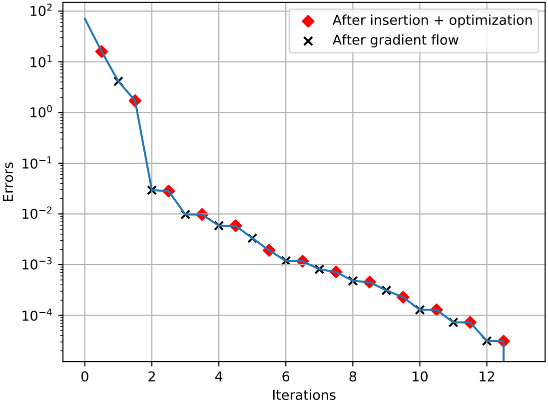

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Outline

General setting

Sparse inverse problems

Solve $$\min_{u \in \mathcal{M}} F(Ku) + \mathcal{G}(u)$$

- $\mathcal{M} = \mathcal{C}^*$ dual space, $\mathcal{C}$ separable, Y Hilbert space
- $K : \mathcal{M} \to Y$, weak*-to-weak continuous and weak*-to-strong continuous in $\mathrm{dom}(\mathcal{G})$
- $F : Y \to \mathbb{R}$ bounded from below, strictly convex, Fréchet differentiable, ∇F Lipschitz on compact sets
- $\mathcal{G} : \mathcal{M} \to [0, \infty]$, convex, weak* lower semi-continuous, positive one-homogeneous, coercive

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What does sparsity mean in this context?

$\rightsquigarrow$ **extremal points**

General representer theorems

Theorem *[B./Carioni '20]*

- If Y is finite-dimensional, then there are sparse solutions of $\min_{u\in\mathcal{M}} \; F(Ku) + \mathcal{G}(u)$, i.e.,

$$u^* = \sum_{i=1}^{N} v_i^* u_i^*, \qquad v_i^* > 0,\; u_i^* \in \operatorname{Extr}(\{\mathcal{G} \leq 1\})$$

- Also see *[Boyer/Chambolle/De Castro/Duval/De Gournay/Weiss '19]*

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Choquet's theorem

Theorem *[Choquet]*

- X locally convex space, $K \subset X$ non-empty, convex, metrizable, compact
- For each $v \in K$, there is a probability measure μ over X concentrated on $\mathrm{Extr}(K)$ with

$$T(v) = \int_X T \, \mathrm{d}\mu \qquad \text{for each } T \in X^*$$

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Lifting of $\mathcal{G}$:

- With $\mathcal{B} = \overline{\mathrm{Extr}(\{\mathcal{G} \leq 1\})}^*$, there is $\mathcal{I} : \mathcal{M}_+(\mathcal{B}) \to \mathrm{dom}(\mathcal{G})$

$$\begin{cases} \mathcal{G}(\mathcal{I}\mu) \leq \|\mu\|_{\mathcal{M}} \text{ for all } \mu \in \mathcal{M}_+(\mathcal{B}) \\ \mathcal{G}(u) = \|\mu\|_{\mathcal{M}} \text{ for } u \in \mathrm{dom}(\mathcal{G}), \mu \in \mathcal{M}_+(\mathcal{B}), \mathcal{I}\mu = u \end{cases}$$

Lifted conditional gradient method

Equivalent problem: $\min_{\mu \in \mathcal{M}_+(\mathcal{B})} F(K\mathcal{I}\mu) + \|\mu\|_{\mathcal{M}}$

- Employ variant of PDAP algorithm $\rightsquigarrow$ fully corrective generalized conditional gradient method (FC-GCG)

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1. Compute $p^n = -K_* \nabla F(Ku^n)$, find $v^n \in \arg\max_{v \in \mathrm{Extr}(\{\mathcal{G} \leq 1\})} \langle p^n, v \rangle$, set $\mathcal{A}_{n+1/2} = \mathcal{A}_n \cup \{v^n\}$
2. Compute solution λ^{n+1} of
$$\min_{\lambda \in [0,\infty[^{\mathcal{A}_{n+1/2}}} F\Big(\sum_{v \in \mathcal{A}_{n+1/2}} \lambda(v) v\Big) + \sum_{v \in \mathcal{A}_{n+1/2}} \lambda(v)$$
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GCG method $\rightsquigarrow$ convergence rate $\mathcal{O}(n^{-1})$

Linear convergence: Assumptions

Uniqueness and sparsity assumption

- For u^* optimal solution, $y^* = Ku^*$ optimal observation, $p^* = -K_*\nabla F(y^*)$ dual certificate:
- F is strongly convex around y^*
- $\langle p^*, u\rangle = 1$ only at u_i^*, $i = 1, \ldots, N$, $p^* < 1$ otherwise on $\mathcal{B}$
- $Ku_1^*, \ldots, Ku_N^*$ are linearly independent

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Non-degeneracy assumption

- $u^* = \sum_{i=1}^N \lambda_i^* u_i^*$ with $\lambda_1^*, \ldots, \lambda_N^* > 0$
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Linear convergence: Result

Theorem *[B./Carioni/Fanzon/Walter '21]*

The FC-GCG method converges linearly under the above assumptions.

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Ingredients for the proof:

- For n large, all $\mathcal{A}_n$ are contained in an appropriate neighborhood of $\{u_1^*, \dots, u_N^*\}$
- Letting $\mu^n = \sum_{v \in \mathcal{A}_n} \lambda^n(v)\delta_v$, relate
$$g(v^n, u_i^*) \leq c\sqrt{r(\mu^n)} \qquad \text{for appropriate } i$$
- Relate $\|K\mathcal{I}(\hat{\mu}^n - \mu^n)\| \leq c\sqrt{r(\mu^n)}$ where $\hat{\mu}^n$ concentrates the mass of μ^n around v^n to v^n
- $r(\mu^{n+1}) \leq qr(\mu^n)$ using a convex combination of μ^n and $\hat{\mu}^n$

(Nontrivial) examples

Minimum effort problems $$\min_{u \in L^\infty(\Omega)} F(Ku) + \alpha\|u\|_\infty$$

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- For $p \in L^1(\Omega)$, we have $\bar{v} \in \arg\max_{v \in \mathrm{Extr}(\{\mathcal{G} \leq 1\})} \langle p, v \rangle$ if
$$\bar{v} = \alpha^{-1} \operatorname{sign}(p)$$
- Non-degeneracy assumption satisfiable for
$g(v, u) = \|v - u\|_1$ and $|\{|p^*| \leq \varepsilon\}| \leq C\varepsilon$

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Trace-norm regularized problems $\min_{\mathcal{U} \geq 0} F(K\mathcal{U}) + \alpha \operatorname{Tr}(\mathcal{U})$

- $\mathrm{Extr}(\{\mathcal{G} \leq 1\}) = \{\alpha^{-1} u \otimes u \mid \|u\| = 1\} \cup \{0\}$
- Solve $\bar{\mathcal{V}} \in \arg\max_{\mathcal{V} \in \mathrm{Extr}(\{\mathcal{G} \leq 1\})} \langle \mathcal{P}, \mathcal{V} \rangle$ via power iteration
- Non-degeneracy assumption satisfiable for rank-1 solutions and $g(\mathcal{V}, \mathcal{U}) = \|\mathcal{V} - \mathcal{U}\|_{\mathrm{HS}}$

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- Introduced rigorous framework for optimal transport regularization of time dependent inverse problems
- Characterization of the extremal points of the Benamou-Brenier regularizer
- Numerical algorithm for dynamic spike reconstruction
- Application to dynamic superresolution
- A generalization to general sparse reconstruction is possible ($\rightsquigarrow$ FC-GCG method)
- Under degeneracy assumption on the solution, the FC-GCG method converges linearly

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